C_o = weight of residual liquid retained initially/volume of residual liquid, lb./cu.ft.
D = molecular diffusivity between the filtrate and wash, sq.ft./min.
d_p = diameter of sphere having same volume as particle constituting the porous medium, ft.
F = fraction of residual liquid washed
I_o = second kind of zero order of the modified Bessel function
I_1 = second kind of first order of the modified Bessel function
k = constant in Equation (41)
l = length of a side channel, ft.
N = rate of transport of filtrate from the blind side channels into wash liquor flowing down the straight channel, lb./(min.)(sq.ft.), of blind side channel cross section
Q = flow rate of wash, cu.ft./min.
R = residual saturation defined as volume of residual liquid per void volume, cu.ft./cu.ft.
s = dummy variable in Equation (31)
t = washing period, min.
v = average velocity of the fluid based on the cross-sectional area of void spaces, ft./min.
V = total volume of the bed, cu.ft.
W = volume of wash liquor throughput/volume of void volume
y = height of packed column, ft.
Y = aDy/v

Greek Letters

$\mu(t, y)$ = concentration of filtrate in the straight channel at time t and y ft. from the top of straight channel, lb. of filtrate/cu.ft. of liquids (mixture of filtrate and wash)
$\rho(\theta, y)$ = concentration of filtrate in the straight channel
ϵ = porosity, volume of voids per total volume of bed, cu.ft./cu.ft.
δ = length of the side channel unfilled with the residual liquid, ft.
$\theta = t - y/v$ = the time at which filtrate is moved from the side channel, min.
β_n = roots of J_o

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Flow of Viscoelastic Fluids Through Porous Media

JOHN C. SLATTERY
Mobil Oil Corporation, Dallas, Texas

Local volume averaging of the equations of continuity and of motion over a porous medium is discussed. For steady state flow such that inertial effects can be neglected, a resistance transformation is introduced which in part transforms the local average velocity vector into the local force per unit volume which the fluid exerts on the pore walls. It is suggested that for a randomly deposited, although perhaps layered, porous structure this resistance transformation is invertible, symmetric, and positive-definite. Finally, for an isotropic porous structure (the proper values of the resistance transformation are all equal and are termed the resistance coefficient) and an incompressible fluid, the functional dependence of the resistance coefficient is discussed with the Buckingham-Pi theorem used for an Ellis model fluid, a power model fluid, a Newtonian fluid, and a Noll simple fluid. Based on the discussion of the Noll simple fluid, a suggestion is made for the correlation and extrapolation of experimental data for a single viscoelastic fluid in a set of geometrically similar porous structures.

Darcy's law, involving a parameter k, termed the *permeability*, was originally proposed as a correlation of experimental data for the flow of an incompressible Newtonian fluid of viscosity μ moving axially with a volume flow rate Q through a cylindrical packed bed of cross section A and length l under the influence of a pressure difference ΔP (*1*, p. 634):

$$\frac{\Delta P}{l} = \frac{\mu}{k}\frac{Q}{A} \tag{1}$$

John C. Slattery is at Northwestern University, Evanston, Illinois.

Equation (1) has suggested for an isotropic porous medium a vector form of Darcy's law (*1*, p. 634):

$$\nabla P + \frac{\mu}{k}\mathbf{V} = 0 \tag{2}$$

A major difficulty of this equation has been that since it was not derived, the average pressure P and average velocity $\mathbf{V}$ were undefined. Whitaker (*2*) has recently derived a generalization of Equation (2) appropriate to an anisotropic porous medium by taking a local average of the equation of motion. In his result, P and $\mathbf{V}$ are local surface averages of pressure and velocity, respectively.

The object of this paper is to develop, by a method considerably different from Whitaker's, an extension of Darcy's law which is appropriate to viscoelastic fluids. [Viscoelastic is used here in the sense that the materials obey neither of the classical linear relations: Newton's law of viscosity and Hooke's law of elasticity. Subclasses of such materials are fluids which show a finite relaxation time (in any one of several senses) and fluids which exhibit normal stress effects in viscometric flows (7, p. 47). The term *viscoelastic* is commonly used in the literature in referring to one of these subclasses.] We begin by discussing in the first section the problem of local volume averaging of the equation of motion as opposed to the local surface averaging explained by Whitaker. In the second section a resistance transformation (the words *transformation* and *tensor* are used interchangeably here) is introduced to describe in part the force per unit volume which the fluid exerts on the (perhaps) layered, porous medium. In the third and fourth sections we deal with an isotropic medium and consider the functional dependence of the resistance parameter by means of the Buckingham-Pi theorem. In the third section we take up two simple empirical models which do not account for normal stress effects or the possible memory of the fluid. In the fourth section we consider the problem of the incompressible Noll simple fluid, currently believed to be a general description of a wide variety of memory fluids. We conclude with discussions of anisotropic porous media and available experimental data.

AN AVERAGE OF THE EQUATIONS OF MOTION AND CONTINUITY

Let us consider a particular point in the porous media and let there be associated with this point a closed surface S, the volume of which is V. We may assume that this point lies in the interior of S, but this is not essential.

Let us take as our system all of the pores which contain fluid in the interior of S and let us make a momentum balance on this system by integrating the stress equation of motion over the volume of this system, $V_{(s)}$:

$$\int_{V_{(s)}} \left\{ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \rho \mathbf{v}\mathbf{v} - \nabla \cdot \mathbf{t} - \rho \mathbf{f} \right\} dV = 0 \quad (3)$$

After interchanging differentiation and integration in the first term (the boundaries of the system are fixed in space) and after applying the divergence theorem to the second and third terms, we have

$$\frac{\partial}{\partial t}\int_{V_{(s)}} \rho \mathbf{v}\, dV + \int_{S_{(s)}} \rho \mathbf{v}\,[\mathbf{v}\cdot\mathbf{n}]\, dS - \int_{S_{(s)}} \mathbf{t} \quad \mathbf{n}\, dS - \int_{V_{(s)}} \rho \mathbf{f}\, dV = 0 \quad (4)$$

Here we denote by $S_{(s)}$ the closed boundary surface of our system composed of the pore walls S_f and the intersections S_e of the pores with S. We may consider S_e as the entrance and exit surfaces of the system.

Let us now associate the surface S with every point in the porous medium by a translation of S. For example, if S is a unit sphere, the center of which coincides with the point considered, we center upon each point in the porous medium a unit sphere. If S is small compared with the average pore diameter, it may enclose only solid or only fluid at many points; if it is large, many pores may intersect with S, the intersections serving as entrances and exits to the fluid system enclosed by S. Consider an arbitrary curve in the porous medium and let s be a parameter such as arc length measured along this curve. We may identify with each point along this curve a system composed of the pores containing fluid enclosed by the surface S and we may ask how the integral over $V_{(s)}$ of some quantity B associated with the fluid changes as a function of s. By the generalized transport theorem (3, p. 347)

$$\frac{d}{ds}\int_{V_{(s)}} B\, dV = \int_{V_{(s)}} \frac{\partial B}{\partial s}\, dV + \int_{S_{(s)}} B\,\frac{d\mathbf{p}}{ds}\cdot \mathbf{n}\, dS \quad (5)$$

Here $\mathbf{p}$ is the position vector. The porous medium is fixed in space and we may think of the closed surface S as being translated (without rotation) along this arbitrary curve through the porous medium. If we denote by $S_{(s)}$ the bounding surface of the fluid which is enclosed by S as a function of arc length s, as we move along this curve that portion of $S_{(s)}$ formed by the pore walls S_f is fixed as a function of s and $d\mathbf{p}/ds = 0$ on S_f. On the intersections of the pores with S [the entrances and exits of $V_{(s)}$], S_e, $d\mathbf{p}/ds$ is uniform and equal to the value of this quantity on the arbitrary curve. In order to establish this latter result, it is essential that S is associated with every point in the porous medium by a *translation* (without rotation) from the original point of definition. We further restrict ourselves to quantities B which are explicit functions of position only, for which $\partial B/\partial s = 0$ (by $\partial B/\partial s$ we mean a derivative with respect to s holding position fixed). This allows us to rewrite Equation (5) as

$$\frac{d\mathbf{p}}{ds}\cdot \nabla \int_{V_{(s)}} B\, dV = \frac{d}{ds}\int_{V_{(s)}} B\, dV = \frac{d\mathbf{p}}{ds}\cdot \int_{S_e} B\mathbf{n}\, ds \quad (6)$$

But since $d\mathbf{p}/ds$ is arbitrary, we conclude that

$$\nabla \int_{V_{(s)}} B\, dV = \int_{S_e} B\mathbf{n}\, dS \quad (7)$$

If B is interpreted as a tensor quantity, we may write as a special case of Equation (7)

$$\nabla \cdot \int_{V_{(s)}} \mathbf{B}\, dV = \int_{S_e} \mathbf{B}\cdot \mathbf{n}\, dS \quad (8)$$

Equation (4) may now be written by using Equation (8) to obtain

$$\frac{\partial}{\partial t}\int_{V_{(s)}} \rho \mathbf{v}\, dV + \nabla \cdot \int_{V_{(s)}} \rho \mathbf{v}\mathbf{v}\, dV = \nabla \cdot \int_{V_{(s)}} \mathbf{t}\, dV + \int_{S_f} \mathbf{t}\cdot \mathbf{n}\, dS + \int_{V_{(s)}} \rho \mathbf{f}\, dV \quad (9)$$

If we divide each term in this equation by V and denote by an overbar an average over V of quantities associated with the fluid, we have

$$\frac{\partial\, \overline{\rho \mathbf{v}}}{\partial t} + \nabla \cdot \overline{\rho \mathbf{v}\mathbf{v}} = \nabla \cdot \overline{\mathbf{t}} + \overline{\rho \mathbf{f}} + \frac{1}{V}\int_{S_f} \mathbf{t}\cdot \mathbf{n}\, dS \quad (10)$$

This is a volume average of the equation of motion in terms of volume-averaged variables.

It will be convenient to consider separately the viscous and pressure forces acting at the entrances and exits of the system $V_{(s)}$. Accordingly, the first term on the right of Equation (10) may be written as

$$\nabla \cdot \overline{\mathbf{t}} = \frac{1}{V}\int_{S_e} \mathbf{t}\cdot \mathbf{n}\, dS = -\nabla \overline{p} + \nabla \cdot \overline{\boldsymbol{\tau}} \quad (11)$$

where the viscous portion of the stress tensor $\boldsymbol{\tau}$ is defined as

$$\boldsymbol{\tau} = \mathbf{t} + p\mathbf{I} \quad (12)$$

The equation of continuity may be volume averaged in a manner similar to that used to obtain the volume-averaged equation of motion, Equation (10). The result is

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot \overline{\rho \mathbf{v}} = 0 \tag{13}$$

THE RESISTANCE TRANSFORMATION

Let us assume that the body force vector per unit mass $\mathbf{f}$ is a constant throughout the system and that it may be represented by a potential Φ:

$$\mathbf{f} = -\nabla\Phi \tag{14}$$

If we restrict ourselves to steady state flows of incompressible fluids such that inertial effects can be neglected, Equations (10) and (11) give

$$\nabla\bar{\mathcal{P}} - \frac{1}{V}\int_{S_f} [\mathbf{t} - \rho\,\phi\,\mathbf{I}] \cdot \mathbf{n}\, dS - \nabla \cdot \bar{\boldsymbol{\tau}} = 0 \tag{15}$$

where the volume-averaged modified pressure $\bar{\mathcal{P}}$ is defined by

$$\bar{\mathcal{P}} = \bar{p} + \rho\Phi$$

It is easy to show that Equation (15) may also be written as

$$\nabla\overline{[\mathcal{P} - p_o]} - \frac{1}{V}\int_{S_f} \{\mathbf{t} + [p_o - \rho\,\Phi]\mathbf{I}\} \cdot \mathbf{n}\, dS - \nabla\,\bar{\boldsymbol{\tau}} = 0 \tag{16}$$

where p_o is a constant ambient or reference pressure.

The last two terms on the left of Equation (16) may be expressed as the result of a resistance transformation $\mathbf{K}$ acting upon the average velocity vector $\mathbf{v}$:

$$-\frac{1}{V}\int_{S_f} \{\mathbf{t} + [p_o - \rho\,\Phi]\mathbf{I}\} \cdot \mathbf{n}\, dS - \nabla \cdot \bar{\boldsymbol{\tau}} = \mathbf{K} \cdot \bar{\mathbf{v}} \tag{17}$$

Equations (16) and (17) combine to give a generalized form of Darcy's law:

$$\nabla\,\overline{[\mathcal{P} - p_o]} + \mathbf{K} \cdot \bar{\mathbf{v}} = 0 \tag{18}$$

Such an equation has been suggested previously on intuitive grounds (*4*). Whitaker's result (*2*) is similar, although his averages are surface averages.

A linear transformation $\mathbf{A}$ on a finite-dimensional vector space is invertible if, and only if, $\mathbf{A} \cdot \mathbf{x} = 0$ implies that $\mathbf{x} = 0$ (*5*, p. 62). It seems reasonable to say that, if in steady state flow the gradient of $\overline{[\mathcal{P} - p_o]}$ is zero, that is, if

$$\mathbf{K} \cdot \bar{\mathbf{v}} = 0$$

and if S is so large as to intersect portions of many pores, then the average velocity $\bar{\mathbf{v}}$ of the fluid is zero. On this basis we will say that $\mathbf{K}$ is invertible.

By the polar decomposition theorem (*6*, p. 841), we may write

$$\mathbf{K} = \mathbf{R} \cdot \mathbf{U} \tag{19}$$

where $\mathbf{R}$ is a uniquely determined orthogonal transformation (a rotation) and $\mathbf{U}$ is a uniquely determined symmetric, positive-definite transformation (the components of a vector in the principal directions of $\mathbf{U}$ are stretched). In what follows we restrict the discussion to porous media and S such that $\mathbf{R} = \mathbf{I}$, the identity transformation. We believe that a randomly deposited, although perhaps layered, porous structure and an S so large as to intersect portions of many pores should meet this condition, since, while it is likely that there are preferred (proper) directions due to layering, it is unlikely that there are preferred rotations. For such a porous structure and such an S it seems reasonable that, if the average velocity at a point in the porous medium is in one of the preferred directions, the resulting force per unit volume on the porous structure (beyond the hydrostatic force and the force due to the ambient pressure) will be in the same preferred direction. If for the moment we neglect the second term on the left of Equation (17) (in making momentum balances for gross systems, it is a common assumption to neglect viscous forces at the entrances and exits of the system), we suggest in this way that for such a porous structure $\mathbf{R} = \mathbf{I}$. That $\mathbf{K}$ is an invertible, symmetric, positive-definite transformation is then a consequence of Equation (19). A positive-definite transformation is such that its proper values are all real, positive, and nonzero (*5*, p. 153). This discussion is consistent with Whitaker's conclusion (*2*) that for anisotropic porous media the resistance transformation $\mathbf{K}$ is not necessarily symmetric.

If as well there are no preferred directions in the porous medium, the three proper values of $\mathbf{K}$ are equal and

$$\mathbf{K} = K\mathbf{I} \tag{20}$$

where the resistance coefficient K is a real, positive, nonzero number. Such a porous body is said to be isotropic.

Before we go further, it is worth noting that $\bar{p}$ is *not* the average pressure at a point in terms of which experimentalists usually think. They are more likely to measure something close to

$$\langle p \rangle \equiv \frac{1}{V_{(s)}}\int_{V_{(s)}} p\, dV = \frac{\bar{p}}{\psi} \tag{21}$$

where ψ is the porosity of the structure. In a static situation, they expect

$$\nabla \langle \mathcal{P} \rangle = \nabla \langle p + \rho\,\Phi \rangle = 0$$

since they know that $\mathcal{P}$ should be a constant p_o throughout the fluid. This is consistent with the static limit of Equation (18):

$$\nabla\overline{[\mathcal{P} - p_o]} = 0$$

SOME SIMPLE MODELS OF FLUID BEHAVIOR

For the rest of this discussion we restrict ourselves to a steady state incompressible flow through an isotropic porous structure such that inertial effects can be neglected. Under these assumptions we have from Equations (18) and (20)

$$\nabla\,\overline{[\mathcal{P} - p_o]} + K\mathbf{v} = 0 \tag{22}$$

One commonly used empirical model which makes no attempt to describe memory effects or normal-stress effects (*7*) is the three-parameter Ellis model (*8*, p. 246) which represents the rate of deformation tensor $\mathbf{d}$ as a function of the extra stress τ:

$$\mathbf{d} = \frac{1}{2\eta_o}\left\{1 + \left[\frac{\tau}{\tau_{1/2}}\right]^{\alpha-1}\right\}\boldsymbol{\tau} \tag{23}$$

where

$$\mathbf{d} = \frac{1}{2}\{\nabla\mathbf{v} + (\nabla\mathbf{v})^T\} \tag{24}$$

and

$$\tau = \sqrt{\frac{1}{2}\, tr\, \boldsymbol{\tau}^2} = \sqrt{\frac{1}{2}\, \tau^{ij}\tau_{ij}} \tag{25}$$

Here η_o, $\tau_{1/2}$, and α are material parameters which are assumed to depend only upon the local thermodynamic state; $(\nabla\mathbf{v})^T$ represents the transpose of $\nabla\mathbf{v}$. One reasonable approach for this model is to say that K should be a function of a characteristic length of the system L, the magnitude of a characteristic velocity, say $\bar{v}$, and the three material parameters in Equation (23):

$$K = F_1(L, \bar{v}, \eta_o, \tau_{1/2}, \alpha) \quad (26)$$

We have not considered the fluid density here, since it does not appear in the restricted form of the equation of motion, Equation (15), [the second term on the left of Equation (15) represents the force per unit volume which the fluid exerts on the porous structure *beyond* the hydrostatic force and the force due to the ambient pressure] and since it does not enter the volume averaged equation of continuity, Equation (13), when ρ is taken to be a constant:

$$\nabla \cdot \bar{\mathbf{v}} = 0 \quad (27)$$

By the Buckingham-Pi theorem (9) Equation (26) can be written in terms of a dimensionless permeability k_o^*

$$K = \frac{\eta_o}{L^2 k_o^*} \quad (28)$$

where

$$k_o^* = f_1\left(\frac{\bar{v}\,\eta_o}{\tau_{1/2} L}, \alpha\right) \quad (29)$$

In Equation (26) one might assume that K is a function of the magnitude of the local averaged pressure gradient, $|\nabla \overline{[\mathcal{P} - p_o]}|$, rather than $\bar{v}$:

$$K = F_2(L, |\nabla \overline{[\mathcal{P} - p_o]}|, \eta_o, \tau_{1/2}, \alpha) \quad (30)$$

The Buckingham-Pi theorem again yields Equation (28) where now

$$k_o^* = f_2\left(\frac{|\nabla \overline{[\mathcal{P} - p_o]}|\, L}{\tau_{1/2}}, \alpha\right) \quad (31)$$

A second alternative to Equation (26) would be to take

$$K = F_3(L, |\nabla \overline{[\mathcal{P} - p_o]}|, \bar{v}, \eta_o, \tau_{1/2}, \alpha) \quad (32)$$

From the Buckingham-Pi theorem we have Equation (28) and

$$k_o^* = f_3\left(\frac{|\nabla \overline{[\mathcal{P} - p_o]}|\, L}{\tau_{1/2}}, \frac{\bar{v}\,\eta_o}{\tau_{1/2} L}, \alpha\right) \quad (33)$$

On the other hand, since K, $|\nabla \overline{\mathcal{P} - p_o]}|$, and $\bar{v}$ are not independent variables in view of Equation (22), Equation (32) does not appear to be an entirely reasonable choice.

A special case of Ellis model fluid is the power model fluid which can be written as

$$\boldsymbol{\tau} = 2\,\eta_1\,\gamma^{n-1}\,\mathbf{d} \quad (34)$$

where

$$\gamma = \sqrt{2\,tr\,\mathbf{d}^2} = \sqrt{2\,d^{ij}d_{ij}} \quad (35)$$

If one assumes by analogy with Equation (26) that

$$K = G_1(L, \bar{v}, \eta_1, n) \quad (36)$$

by the Buckingham-Pi theorem we conclude that

$$K = \frac{\bar{v}^{n-1}\,\eta_1}{L^{n+1}\,k_1^*} \quad (37)$$

where

$$k_1^* = g_1(n) \quad (38)$$

If one assumes by analogy with Equation (30) that

$$K = G_2(L, |\nabla \overline{[\mathcal{P} - p_o]}|, \eta_1, n) \quad (39)$$

we have from the Buckingham-Pi theorem that

$$K = \frac{|\nabla \overline{[\mathcal{P} - p_o]}|^{\frac{n-1}{n}}\,\eta_1^{\frac{1}{n}}}{L^{\frac{n+1}{n}}\,k_2^*} \quad (40)$$

where

$$k_2^* = g_2(n) \quad (41)$$

A special case of the power model fluid is the Newtonian fluid

$$\boldsymbol{\tau} = 2\,\mu\mathbf{d} \quad (42)$$

By replacing η_1 by μ and letting n go to unity, we have from either Equations (37) or (40)

$$K = \frac{\mu}{L^2\,k^*} \quad (43)$$

where k^* is a constant. This result, together with Equation (22), yields a familiar form of Darcy's law.

THE NOLL SIMPLE FLUID

Let us focus our attention on a particular material particle in a body which has been undergoing a deformation as a function of time. Let $\boldsymbol{\xi}$ be the place at time $t + s$ of that material particle which at time t occupies the place $\mathbf{x}$:

$$\boldsymbol{\xi} = \boldsymbol{\chi}_{(t)}(\mathbf{x}, t + s) \quad (44)$$

We call $\boldsymbol{\chi}_{(t)}$ the relative deformation function. By means of Equation (44) we may describe the motion which took place in the material at all prior times $t + s$ $(-\infty < s < 0)$. The gradient of the relative deformation function is called the relative deformation gradient and it is denoted by $\mathbf{F}_{(t)}$:

$$\mathbf{F}_{(t)} \equiv \nabla\,\boldsymbol{\chi}_{(t)}(\mathbf{x}, t + s) \quad (45)$$

The relative right Cauchy-Green strain tensor is defined as

$$\mathbf{c}(t + s) = \mathbf{F}^T{}_{(t)} \cdot \mathbf{F}_{(t)} \quad (46)$$

where $\mathbf{F}^T{}_{(t)}$ represents the transpose of $\mathbf{F}_{(t)}$.

All constitutive equations for stress which have found any success at all have satisfied what Noll terms the principle of determinism for stress (the stress in the body is determined by the history of the motion of that body) and the principle of local action (in determining the stress at the given material particle, the motion outside an arbitrary neighborhood of the particle may be disregarded) (*10*, p. 56). A *simple material* (*10*, p. 61) is defined by Noll to be one for which the stress $\mathbf{t}$ at $\mathbf{x}$ and at time t is determined by the history of the relative deformation gradient $\mathbf{F}_{(t)}$ for the material which is an arbitrarily small neighborhood of $\mathbf{x}$ at time t. He defines an incompressible *simple fluid* (*10*, p. 81) as one for which stress $\mathbf{t}$ at $\mathbf{x}$ and at time t is specified within an indeterminate pressure p by the history of the relative right Cauchy-Green strain tensor for the material which is within an arbitrarily small neighborhood of $\mathbf{x}$ at time t:

$$\mathbf{t} + p\mathbf{I} = \frac{\mu_o}{t_o} \overset{o}{\underset{\sigma=-\infty}{\mathcal{H}}} [\mathbf{c}(t + t_o\sigma)] \quad (47)$$

Here we follow Truesdell's discussion of the dimensional invariance of the definition of a simple material (*11*). The quantity $\overset{o}{\underset{\sigma=-\infty}{\mathcal{H}}}$ is a dimensionally invariant tensor-valued functional, that is, an operator which maps tensor-valued functions into tensors. The constants μ_o and t_o are, respectively, a characteristic time of the fluid. Although Truesdell suggests specific definitions for these quantities, for our discussion it is not necessary to make particular choices.

Equation (47) suggests by analogy with Equation (26) the following functional dependence for K in Equation (22):

$$K = H_1(L, \bar{v}, \mu_o, t_o) \quad (48)$$

For simplicity we have not included in this relationship the (perhaps) many dimensionless parameters in terms of which any particular functional $\overset{o}{\underset{\sigma=-\infty}{\mathcal{H}}}$ would be represented. By the Buckingham-Pi theorem we find that

$$K = \frac{\mu_o}{L^2 \kappa^*} \quad (49)$$

where the dimensionless permeability κ^* has the following functional dependence:

$$\kappa^* = h_1 \left(\frac{\bar{v}\, t_o}{L} \right) \quad (50)$$

If we write instead by analogy with Equation (30)

$$K = H_2 (L, | \nabla \overline{[\mathcal{P} - p_o]} |, \mu_o, t_o) \quad (51)$$

we conclude that Equation (49) holds where now

$$\kappa^* = h_2 \left(\frac{| \nabla \overline{[\mathcal{P} - p_o]} | L\, t_o}{\mu_o} \right) \quad (52)$$

The most useful correlation of experimental data is one valid for all fluids of a given class. This permits the prediction of the results for one fluid of this class in a given geometry on the basis of previous experiments with perhaps another fluid of the same class in a geometrically similar situation. To be able to do this for viscoelastic fluids it would be necessary to have a specific form of the functional $\overset{o}{\underset{\sigma=-\infty}{\mathcal{H}}}$ applicable to all fluids of a certain class. Although many specific constitutive equations have been proposed, such as the Ellis model and the power model discussed previously, at best they have been shown to be useful in describing a portion of the stress-deformation behavior [the viscosity function in viscometric flows (*7*)] for a few fluids over a limited range of stress.

The next most useful scheme for correlating experimental data allows one to predict the behavior of a particular fluid in a given geometry on the basis of data for the same fluid in a geometrically similar situation. This approach to scale-up for viscoelastic fluids has been discussed previously with turbulent flow in an infinitely long tube taken as an illustration (*12*). For a given fluid, that is holding fixed μ_o, t_o, and all of the dimensionless parameters necessary to describe a specific functional $\overset{o}{\underset{\sigma=-\infty}{\mathcal{H}}}$ in Equation (47), in geometrically similar porous media we should be able to correlate data (and therefore extrapolate a limited amount of data) by regarding L^2K as a function of a single parameter as suggested either by Equations (49) and (50) or by Equations (49) and (52):

$$L^2K = \kappa_1 (\bar{v}/L) \quad (53)$$

or

$$L^2K = \kappa_2 (L \,| \nabla \overline{[\mathcal{P} - p_o]} |) \quad (54)$$

Here we take advantage of the fact that since we are dealing with a single fluid, it is not necessary to have values for the characteristic viscosity μ_o and the characteristic time t_o.

ANISOTROPIC POROUS MEDIA

Referring to the discussion which follows Equation (18), let us restrict ourselves to an anisotropic porous medium such that the resistance transformation **K** is invertible, symmetric, and positive-definite. We have already suggested that a randomly deposited, although perhaps layered, porous structure should meet this condition. For simplicity, let us assume that the proper vectors (or principal directions) of **K** are invariant throughout the porous bed. In a rectangular Cartesian coordinate system, the axes of which coincide with the principal directions, the i^{th} component of Equation (18) is

$$\frac{\partial \overline{[\mathcal{P} - p_o]}}{\partial z_i} + K_{ii} \bar{v}_i = 0 \quad (\text{no sum on } i; \quad i = 1,2,3) \quad (55)$$

Here the rectangular Cartesian coordinates are $z_i (i = 1,2,3)$ and v_i is the i^{th} component of the volume-averaged velocity vector $\bar{\mathbf{v}}$. The quantity K_{ii} is the i^{th} element in the diagonalized matrix for **K**.

We may now discuss the functional dependence of K_{ii} using the Buckingham-Pi theorem in much the same way as we did for K in the preceding two sections; we merely replace K by K_{ii} in those discussions. For example, arguments analogous to those leading up to Equations (53) and (54) suggest that we might correlate data (and therefore extrapolate a limited amount of data) for a single viscoelastic fluid by regarding L^2K_{ii} as a function of a single parameter, either

$$L^2K_{ii} = \beta_{(i)} (\bar{v}/L) \quad (\text{no sum on } i) \quad (56)$$

or

$$L^2 K_{ii} = \gamma_{(i)} (L \,| \nabla \overline{[\mathcal{P} - p_o]} |) \quad (\text{no sum on } i) \quad (57)$$

In general, no relation between the three functions $\beta_{(i)} (i = 1,2,3)$ [or the three functions $\gamma_{(i)} (i = 1,2,3)$] should be expected.

COMPARISON WITH EXPERIMENT

Let us now consider what we believe to be only a slightly idealized version of the most common packed-bed experiment. We call it steady state, rectilinear flow of an incompressible fluid through a uniformly packed bed. It is not necessary to describe in detail our closed surface S used in averaging; it is sufficient to repeat our earlier caution (in discussing the Resistance Transformation) that S should be so large as to intersect portions of many pores. By rectilinear flow we mean that at each point in the packed bed there is only one nonzero component of the volume-averaged velocity $\bar{\mathbf{v}}$ and this axial component of velocity does not depend upon position in a plane which is orthogonal to the direction of flow. The volume-averaged equation of continuity, Equation (13), assures us that for an incompressible fluid the axial component of velocity cannot be a function of axial position. By a uniformly packed bed we mean that the form of any relationship describing the resistance coefficient, such as Equations (28) and (31), is independent of position in the packed bed. In this way we conclude from Equation (22) that $\nabla \overline{[\mathcal{P} - p_o]}$ has only one nonzero component, an axial component, and that this axial component is independent of axial position.

Sadowski and Bird (*13*) and Sadowski (*14*) discuss an experiment which we believe can be reasonably described as steady state, rectilinear flow of an incompressible Ellis model fluid* through a uniformly packed bed. If we recognize that superficial velocity is proportional to our magnitude volume-averaged velocity $\bar{v}$ and if we identify our $| \nabla \overline{[\mathcal{P} - p_o]} |$ with their pressure gradient, we may easily show that Sadowski and Bird's Equation (15) (see also Sadowski's Figures 4, 5, and 6) is of the form of Equations (22), (28), and (31). They have an added dependence upon void fraction, since they did not limit themselves to geometrically similar porous media.

The correlation presented by Sadowski (*14*) in his Figure 7 is a particular case of Equations (22), (28), and (33), if we forget about his added dependence upon void fraction. As mentioned previously, since k, $| \nabla \overline{[\mathcal{P} - p_o]} |$, and $\bar{v}$ are dependent through Equation (22), Equation (33) does not appear to be entirely reasonable. We shall return to this point shortly.

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* Sadowski and Bird (13) view the Ellis model as an empirical curve-fitting equation for the viscosity function which arises in viscometric flows of incompressible Noll simple fluids (7, p. 27). But in the context of their development of a model for flow through a packed bed [a nonviscometric flow (7, p. 29] it appears that they have used the more traditional interpretation of the Ellis model given here.

Christopher and Middleman (*15*) considered the flow of a power model fluid through a bundle of capillary tubes. Except for an additional dependence upon void fraction, their Equation (7) is a special case of Equations (22), (40), and (41) applied to a steady state, rectilinear flow of an incompressible power model fluid through a uniformly packed bed.

Equation (22) together with either Equation (53) or Equation (54) may be used to correlate data for steady state, rectilinear flow of incompressible viscoelastic fluids through a uniformly packed bed without making a decision as to the specific constitutive equation required to represent the behavior of the material. All one must assume is that the material is a Noll simple material. For example, if we interpret the variables of Sadowski and Bird (*13*) as we discussed above, their Equation (15) is of the form

$$\frac{LK}{\overline{v}} = \frac{1}{L\overline{v}} f(L \mid \nabla \overline{[\mathcal{P} - p_o]} \mid) \tag{58}$$

Since this is a modification of Equation (54), we see that Equation (54) would be just as successful as their Equation (15) in correlating data *for any particular fluid*. Rather than the straight lines obtained in Sadowski's Figures 4, 5, and 6, (*14*), Equation (54) would give a curve; but the degree of correlation would be essentially the same.

The data correlation presented by Sadowski (*14*) in his Figure 7 attempts to account for elastic effects. We feel that any effects due to the memory of the material have been rather broadly accounted for in Equations (53) and (54) by basing their development upon the Noll simple fluid. Yet, in order to reconcile Equation (54) with his Figure 7, one must assume that K, $\overline{v}$, and $\mid \nabla \overline{[\mathcal{P} - p_o]} \mid$ are independent variables in Equation (51). This does not appear to be consistent with Equation (22). From a different point of view, the dependence upon density should drop out in his correlations if inertial effects are negligible; this is the case in his Figures 4 and 5. Since the data form a curve in his Figure 6, the dependence upon density does not drop out, suggesting that inertial effects may be important as his X (which is proportional to a Reynolds number) increases.

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NOTATION

$\mathbf{d}$ = rate of deformation tensor defined by Equation (24)
d^{ij}, d_{ij} = contravariant and covariant components of the rate of deformation tensor $\mathbf{d}$
$\mathbf{f}$ = body force vector per unit mass
$\mathbf{I}$ = identity transformation
K = resistance coefficient used to represent a proper value of the resistance transformation when all three proper values are identical
$\mathbf{K}$ = resistance transformation as defined by Equation (17)
L = characteristic length of the porous structure, such as $V_{(s)}/S_f$
n = parameter in the power model, Equation (34)
$\mathbf{n}$ = unit normal vector outwardly directed with respect to some closed surface such as $S_{(s)}$
p = pressure
$\mathbf{p}$ = position vector
$\overline{\mathcal{P}}$ = modified pressure $= \overline{p} + \rho\Phi$
$|\nabla\overline{\mathcal{P}}|$ = magnitude of the volume-averaged modified pressure gradient
$\mathbf{R}$ = orthogonal transformation or rotation
s = arc length along an arbitrary curve at Equation (5); time relative to time t at Equation (44)
S = arbitrary closed surface which is associated with every point in the porous medium
$S_{(s)}$ = closed bounding surface of $V_{(s)}$
S_e = entrance and exit portions of $S_{(s)}$
S_f = impermeable or fixed portions of $S_{(s)}$; that portion of $S_{(s)}$ composed of pore walls
t = time
$\mathbf{t}$ = stress tensor
t_o = characteristic time of incompressible Noll simple fluid; left unspecified here
t_r = trace of the operator which follows; for example, $t_r\,\mathbf{A} = A^i{}_i$
$\mathbf{U}$ = symmetric, positive-definite transformation
$\mathbf{v}$ = velocity vector
$V_{(s)}$ = volume of the pores containing fluid which are enclosed by the surface S
$\overline{v}$ = magnitude of the volume-averaged velocity vector
V = volume of S

Greek Letters

α = parameter in the Ellis model, Equation (23)
η_o = parameter in the Ellis model, Equation (23)
η_1 = parameter in the power model, Equation (34)
μ = viscosity of a Newtonian fluid
μ_o = characteristic viscosity of incompressible Noll simple fluid; left unspecified here
ρ = density
σ = used in Equation (47) and defined as s/t_o
$\boldsymbol{\tau}$ = extra stress as defined by Equation (12)
$\tau_{1/2}$ = parameter in the Ellis model, Equation (23)
τ^{ij}, τ_{ij} = contravariant and covariant components of the extra stress tensor τ
Φ = potential in terms of which the body force is represented; defined by Equation (14)
ψ = porosity

Special Symbols

— = overbar indicates an average over the volume V of quantities associated with the fluid
$< >$ = average over the volume $V_{(s)}$ of quantities associated with the fluid

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